UNIT - 02

21/07/2025 EXPONENTIAL DISTRIBUTION Monday

(Imp) the time between occurrences of successive events in a Poisson Process where the events occur

Definition:- A continuous random variable X in the interval $(0,\infty)$ as the probability density function
$$f(x) = \begin{cases} \theta e^{-\theta x}, & x \geq 0;\ \theta > 0 \\ 0 & ; \text{otherwise} \end{cases}$$
is called exponention distribution with parameter θ. It is usually denoted by $x \sim E(\theta)$

(Imp) independently and at a constant average rate

* **Distribution function**:- we know that distribution function it is denoted by $F(x)$ and is defined by as

$$F(x) = P(X = x)$$
$$= \int_0^x f(x)dx$$
$$= \int_0^x \theta e^{-\theta x} dx$$
$$= \theta \int_0^x e^{-\theta x} dx$$
$$= \theta \left[\frac{e^{-\theta x}}{-\theta}\right]_0^x$$
$$= \theta \left[\frac{e^{-\theta x} - e^{-\theta(0)}}{-\theta}\right]$$
$$= -\left[e^{-\theta x} - e^0\right]$$
$$= -\left[e^{-\theta x} - 1\right]$$

(i.e.) If a customer arrives at a shop every 10 minutes on average, then the waiting time between two customers follows exponential distribution

$$\theta = \frac{1}{10}$$

$$\boxed{F(x) = 1-e^{-\theta x}}$$

$\ast$ **Moments** :- we know that r^{th} moment about origin is denoted by μ_r' and is defined as

$$\mu_r' = E[x^r]$$

$$= \int x^r f(x)\,dx$$

$$= \int_0^\infty x^r \theta . e^{-\theta x} dx$$

$$= \theta \int_0^\infty x^r e^{-\theta x} dx$$

$$= \theta \int_0^\infty x^{r+1-1} e^{-\theta x} dx \qquad [\because \angle = \text{Small gamma}]$$

$$= \theta \frac{\angle(r+1)}{\theta^{r+1}} \qquad \left[\int_0^\infty x^{\lambda-1} e^{-a x} dx = \frac{\angle(\lambda)}{a^\lambda}\right]$$

$$= \frac{\theta r!}{\theta^{r+1}} = \frac{\theta r!}{\theta^r \theta} \qquad \angle(r+1) = r!$$

$$\boxed{\mu_r' = \frac{r!}{\theta^r}} \qquad \neq \frac{r!}{\theta^r}$$

If $r=1$

$$\mu_1' = \frac{1!}{\theta} = \frac{1}{\theta}$$

$$\boxed{\mu_1' = \frac{1}{\theta}} \quad \text{— mean}$$

If $r=2 \Rightarrow \mu_2' = \frac{2!}{\theta^2}$

$$\boxed{\mu_2' = \frac{2}{\theta^2}}$$

If $r=3$

$$\mu_3' = \frac{3!}{\theta^3}$$

$$\boxed{\mu_3' = \frac{6}{\theta^3}}$$

If $r=4$ $\quad \mu_4' = \frac{4!}{\theta^4}$

$$\boxed{\mu_4' = \frac{24}{\theta^4}}$$

* Central moments :-

$$\mu_1 = 0$$

VImp $\mu_2 = \mu_2' - (\mu_1')^2$

$$= \frac{2}{\theta^2} - \frac{1}{\theta^2} = \frac{2-1}{\theta^2}$$

$$\boxed{\mu_2 = \frac{1}{\theta^2}}$$ —variance

$$\mu_3 = \mu_3' - 3\mu_2'\mu_1' + 2(\mu_1')^3$$

$$= \frac{6}{\theta^3} - 3\left(\frac{2}{\theta^2}\right)\left(\frac{1}{\theta}\right) + 2\left(\frac{1}{\theta^3}\right)$$

$$= \frac{6-6+2}{\theta^3}$$

$$\boxed{\mu_3 = \frac{2}{\theta^3}}$$

$$\mu_4 = \mu_4' - 4\mu_3'\mu_1' + 6\mu_2'\mu_1'^2 - 3\mu_1'^4$$

$$= \frac{24}{\theta^4} - \frac{24}{\theta^4} + \frac{12}{\theta^4} - \frac{3}{\theta^4}$$

$$\boxed{\mu_4 = \frac{9}{\theta^4}}$$

Skewness:

$$\beta_1 = \frac{\mu_3^2}{\mu_2^3} = \frac{\left(\frac{2}{\theta^3}\right)^2}{\left(\frac{1}{\theta^2}\right)^3} = \frac{4}{\theta^6} \times \frac{\theta^6}{1} = 4$$

$$\boxed{\beta_1 = 4}$$

$$\beta_2 = \frac{\mu_4}{\mu_2^2} = \frac{9}{\theta^4} \times \frac{\theta^4}{1}$$

$$\boxed{\beta_2 = 9}$$

Kurtosis:

$$\gamma_1 = \sqrt{\beta_1} = \sqrt{4} = 2 \Rightarrow \boxed{\gamma_1 = 2}$$

$$\gamma_2 = \beta_2 - 3 = 9 - 3$$

$$\boxed{\gamma_2 = 6}$$

∴ Exponential Distribution is A Lepto kurtic

* M.G.F of Exponential Distribution :- we know that M.G.F is denoted by $M_x(t)$ is denoted and is defined as $M_x(t) = E[e^{tx}]$

$$= \int_0^\infty e^{tx} f(x)\,dx$$

$$= \int_0^\infty \theta e^{-\theta x} e^{tx}\,dx$$

Scanned with OKEN Scanner

$$= \theta \int_0^{\infty} e^{tx-\theta x}\,dx$$

$$= \theta \int_0^{\infty} e^{x(t-\theta)}\,dx$$

$$= \theta \int_0^{\infty} e^{-x(\theta-t)}\,dx$$

$$= \theta \left[\frac{e^{-x(\theta-t)}}{-(\theta-t)} \cancel{dx} \right]_0^{\infty}$$

$$= \theta \left[\frac{e^{-\infty(\theta-t)} - e^{-0(\theta-t)}}{-(\theta-t)} \right]$$

$$= \theta \frac{[0-1]}{-(\theta-t)}$$

$$\boxed{M_X(t) = \frac{-\theta}{-(\theta-t)}}$$

$$= \frac{1}{\frac{\theta-t}{\theta}} = \frac{1}{\frac{\theta}{\theta}-\frac{t}{\theta}}$$

$$= \frac{1}{1-\frac{t}{\theta}}$$

$$\boxed{M_X(t) = \left(1-\frac{t}{\theta}\right)^{-1}}$$ //

Imp

* <u>Moments</u> <u>through</u> <u>M.G.F</u>: We know that

$$M_X(t) = \left(1-\frac{t}{\theta}\right)^{-1}$$

$$(1-x)^{-1} = 1+x^2+x^3+x^4+\ldots\ldots$$

$$1+\mu_1' t + \mu_2' \frac{t^2}{2!} + \mu_3' \frac{t^3}{3!} + \mu_4' \frac{t^4}{4!} + \ldots\ldots$$

$$= 1 + \frac{t}{\theta} + \frac{t^2}{\theta^2} + \frac{t^3}{\theta^3} + \cdots$$

Comparing coefficients of t, t^2, t^3, t^4 on both sides

$\boxed{\mu_1' = \frac{1}{\theta}}$,

$\frac{\mu_2'}{2!} = \frac{1}{\theta^2}$

$\boxed{\mu_2' = \frac{2}{\theta^2}}$,

$\frac{\mu_3'}{3!} = \frac{1}{\theta^3}$

$\boxed{\mu_3' = \frac{6}{\theta^3}}$,

$\frac{\mu_4'}{4!} = \frac{1}{\theta^4}$

$\boxed{\mu_4' = \frac{24}{\theta^4}}$

Imp

* C.F of Exponential Distribution :- we know that C.F is denoted by $\phi_x(t)$ and is defined as

$$\phi_x(t) = E[e^{itx}]$$

$$= \int_0^\infty e^{itx} f(x)\, dx$$

$$= \int_0^\infty e^{itx}\, \theta \cdot e^{-\theta x}\, dx$$

$$= \theta \int_0^\infty e^{(it-\theta)x}\, dx$$

$$= \theta \left[\frac{e^{(it-\theta)x}}{it-\theta}\right]_0^\infty$$

$$= \theta \left[\frac{e^{-\infty(it-\theta)} - e^{0(it-\theta)}}{it-\theta}\right]$$

$$= \theta \left[\frac{0-1}{it-\theta}\right] = \frac{-\theta}{-(\theta - it)}$$

$$= \frac{1}{\left(\frac{\theta}{\theta} - \frac{it}{\theta}\right)}$$

$$\boxed{\phi_X(t) = \left(1 - \frac{it}{\theta}\right)^{-1}}$$ //

* C.G.F of exponential Distribution :- we know that C.G.F is denoted by $K_X(t)$ and is denoted as

$$K_X(t) = \log [M_X(t)]$$

$$= \log \left[1 - \frac{t}{\theta}\right]^{-1}$$

$$K_1 t + K_2 \frac{t^2}{2!} + K_3 \frac{t^3}{3!} + K_4 \frac{t^4}{4!} + \dots$$

$$= \frac{t}{\theta} + \frac{t^2}{2\theta^2} + \frac{t^3}{3\theta^3} + \frac{t^4}{4\theta^4} + \dots$$

comparing coefficients of t, t^2, t^3, t^4 on both sides

$$K_1 = \frac{1}{\theta} = \mu_1' = \text{Mean}$$

$$\frac{K_2}{2!} = \frac{1}{2\theta^2} \Rightarrow K_2 = \frac{1}{\theta^2} \Rightarrow \text{Variance}$$

$$\frac{K_3}{3!} = \frac{1}{3\theta^3} \Rightarrow \frac{6}{3\theta^3} = \frac{2}{\theta^3}$$

$$K_3 = \boxed{\frac{2}{\theta^3} = \mu_3}$$ //

$$\frac{K_4}{4!} = \frac{1}{4\theta^4}$$

$$\Rightarrow \frac{K_4}{24} = \frac{1}{4\theta^4} \Rightarrow K_4 = \frac{6}{\theta^4}$$

$\mu_4 = k_4 + 3k_2^2$

$= \frac{6}{\theta^4} + 3\left(\frac{1}{\theta^2}\right)^2 = \frac{9}{\theta^4}$

$\boxed{\mu_4 = \frac{9}{\theta^4}}$,,

*** Lack of Memory property of Exponential Distrib**

Statement :- If X has an exponential distribut for each & every constant $a \geq 0$ then

$P(Y \leq x / X \geq a) = P(X \leq x)$

where $Y = X - a$

we know that probability density function of exponential distribution is $f(x) = \theta e^{-\theta x}$

proof :- $P(Y \leq x / X \geq a) = \frac{P(Y \leq x \cap X \geq a)}{P(X \geq a)}$

$= \frac{P(X - a \leq x \cap X \geq a)}{P(X \geq a)}$

$= \frac{P(X \leq x + a \cap X \geq a)}{P(X \geq a)}$

$= \frac{P(x + a \geq X \geq a)}{P(X \geq a)}$

$P(Y \leq x / X \geq a) = \frac{P(a \leq X \leq x + a)}{P(X \geq a)}$ —— (1)

Consider L.H.S

$$p(a \le X \le x+a) = \int f(x)\,dx$$

$$= \int_a^{x+a} \theta \cdot e^{-\theta x} dx$$

$$= \theta \int_a^{x+a} e^{-\theta x} dx$$

$$= \theta \left[\frac{e^{-\theta x}}{-\theta}\right]_a^{x+a}$$

$$= - \left[e^{-\theta(x+a)} - e^{-\theta a}\right]$$

$$= -\left[e^{-\theta x - \theta a} - e^{-\theta a}\right]$$

$$= -e^{-\theta x - \theta a} + e^{-\theta a}$$

$$\boxed{p(a \le X \le x+a) = e^{-\theta a}\left[1 - e^{-\theta x}\right]}$$ //

Now consider

$$P(X \ge a) = \int_a^{\infty} f(x)\,dx.$$

$$= \int_a^{\infty} \theta \cdot e^{-\theta x} dx$$

$$= \theta \int_a^{\infty} e^{-\theta x} dx = \theta \left[\frac{e^{-\theta x}}{-\theta}\right]_a^{\infty}$$

$$= -\left[e^{-\infty a} - e^{-\theta a}\right]$$

$$= +e^{-\theta a}$$

from ① $p(Y \le x / X \ge a) = \dfrac{p(a \le X \le x+a)}{P(X \ge a)}$

Scanned with OKEN Scanner

$$= \frac{e^{-\theta a}\left[1-e^{-\theta x}\right]}{e^{-\theta a}}$$

$$= 1-e^{-\theta x} \quad \text{--- (2)}$$

and from R.H.S

$$P(X \le x) = \int_0^x f(x)\,dx$$

$$= \int_0^x \theta e^{-\theta x}\,dx$$

$$= \theta \int_0 e^{-\theta x}\,dx$$

$$= \theta \left[\frac{e^{-\theta x}}{-\theta}\right]_0^x$$

$$= -\left[e^{-\theta x} - e^{\theta(0)}\right]$$

$$= -\left[e^{-\theta x} - 1\right]$$

$$P(X \le x) = 1-e^{-\theta x} \quad \text{--- (3)}$$

from eq (2) and (3)

$$\boxed{P(Y \le x / X \ge a) = P(X \le x)}$$

S.M. Shamsi
25/07